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О НЕКОТОРЫХ КРИТЕРИЯХ ПРЕДЕЛЬНОГО СОСТОЯНИЯ В МЕХАНИКЕ ТВЕРДОГО ТЕЛА

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Предельные состояния поведения материала могут быть различными. В литературе предельные состояния определяются, в частности, как переход от упругого к неупругому диапазону (например, предел текучести), а также как потеря прочности. Для описания предельных состояний используются понятия напряжения, деформации, а также энергии и мощности. Доминирующими до сих пор являются модели, основанные на напряжениях. Одним из наиболее важных вопросов здесь является формулировка гипотез эквивалентности, которые позволяют сравнивать материальные параметры, которые сами являются скалярными величинами, с соответствующими эквивалентными величинами на основе тензора напряжений. В статье представлены некоторые гипотезы эквивалентности, возможная классификация и открытые вопросы.

Ключевые слова: критерии предельного состояния, эквивалентные напряжения, основные испытания материалов

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1. Motivation. Since the time of Galilei, Hooke and others, the question of limit behavior has been in the context of modeling material behavior. External loadings cause reactions in the material which, within the framework of phenomenological approaches, can be expressed by stresses, strains, but also by energy or power. From a certain stress level, depending on the material and external conditions, one can observe phenomena of material failure such as yielding or brittle fracture [1, 2]. The description of failure modes is complicated, especially since there is no rational approach to it. The reasons for this include the different failure modes (plastic flow, creep, fatigue, fracture, etc.) as well as the multitude of influencing factors (stress level, temporal change of the stresses, temperature, environmental influences, etc.). Numerous experimental studies have shown that the stress level and the type of loading have a particular influence with respect to brittle or ductile failure [3–10], among others. Failure states in the narrower sense are all critical states immediately before fracture (brittle materials) or the transition to plastic flow (ductile materials) [39]. The models of the limit behavior are connected with the formulation of limit criteria (e.g. in the stress space). In the first case (brittle failure) one speaks of strength criteria, in the second case (plastic flow) of yield or plasticity criteria. A discussion of failure criteria in the broader sense (consideration of material damage, failure due to cyclic loading) was made in the following references [11–16], among others.

The formulation of phenomenological theories of failure is controversial. In addition to the view that failure cannot, in principle, be described within the framework of phenomenological models, one also finds special articles and monographs which demonstrate the possibilities of phenomenological concepts and in which failure criteria are formulated primarily for isotropic materials [17–28]. Depending on the number of basic experiments (or the parameters to be determined), criteria can be classified [28]. Thus, 1-parameter criteria, 2-parameter criteria, etc. are obtained.

In the present paper we discuss phenomenological limit criteria. The following limiting assumptions are made:

• The considered materials are macroscopically isotropic and homogeneous.

• The state of stress is characterized exclusively by the resulting state of stress in the material, non-mechanical influencing factors are not taken into account.

• The material is monotonically loaded.

• Only plastic flow or loss of strength (fracture) are accepted as possible failure states.

The following discussions cover some classic criteria, although the goal is not to give a score on the criteria. The main idea is to introduce the criteria and make a possible classification. Before formulation the criteria, some elementary mathematical basics must be introduced.

2. Mathematical Basics. Restricting ourself to stress-based limit criteria for isotropic materials and static monotonous loading, the stress tensor invariants and principal stresses, among others, are of particular importance. Besides the stress tensor invariants, the stress deviator invariants, but also other quantities sometimes should be included in the formulation of criteria. The mathematical fundamentals are briefly given below.

2.1. Principal Stresses and Directions. Let us introduce the stress tensor $\boldsymbol{\sigma}$ [1]. The eigenvalue problem for this tensor is defined by the following equation

$$\boldsymbol{\sigma} \cdot \mathbf{a} = \lambda \mathbf{a} \quad \text{with} \quad \mathbf{a} \neq \mathbf{0} \tag{2.1.1}$$

$\mathbf{a}$ and λ are the eigenvector and the eigenvalue of the problem, $\cdot$ denotes the scalar product. The solution can estimated from

$$(\boldsymbol{\sigma} - \lambda \mathbf{I}) \cdot \mathbf{n} = \mathbf{0} \quad \text{with} \quad \mathbf{n} \cdot \mathbf{n} = 1 \tag{2.1.2}$$

$\mathbf{n}$ is a orthonormal eigenvector and $\mathbf{I}$ is the sekond rank unit tensor. The characteristic equation yields the eigenvalues

$$\det(\boldsymbol{\sigma} - \lambda \mathbf{I}) = 0 \tag{2.1.3}$$

As shown, for example [29], in the case of symmetric stress tensor one distinguish three real solutions following from

$$\lambda^3 - \lambda^2 J_1(\boldsymbol{\sigma}) + \lambda J_2(\boldsymbol{\sigma}) - J_3(\boldsymbol{\sigma}) = 0 \tag{2.1.4}$$

with the principal invariants

$$\begin{aligned} J_1(\boldsymbol{\sigma}) &= \boldsymbol{\sigma} \cdot\cdot \mathbf{I} = \operatorname{tr}\boldsymbol{\sigma} \\ J_2(\boldsymbol{\sigma}) &= \frac{1}{2}[J_1^2(\boldsymbol{\sigma}) - J_1(\boldsymbol{\sigma}^2)] \\ J_3(\boldsymbol{\sigma}) = \det\boldsymbol{\sigma} &= \frac{1}{3}\left[J_1(\boldsymbol{\sigma}^3) + 3J_1(\boldsymbol{\sigma})J_2(\boldsymbol{\sigma}) - J_1^3(\boldsymbol{\sigma})\right] = \\ &= \frac{1}{3}\left[J_1(\boldsymbol{\sigma}^3) - \frac{1}{2}J_1(\boldsymbol{\sigma}^2)J_1(\boldsymbol{\sigma}) + \frac{1}{6}J_1^3(\boldsymbol{\sigma})\right] \end{aligned} \tag{2.1.5}$$

Equation (2.1.) has three solutions (principal stresses) $\sigma_{\mathrm{I}} \geq \sigma_{\mathrm{II}} \geq \sigma_{\mathrm{III}}$:

• three distinguished solutions

• one distinguished solution and one double solution and

• one triple solution.

In the first case one can identify three principal directions, in the second case one principal direction and in the third case no principal direction.

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[1] Here we use the classical stress tensor. The components of this stress tensor are the engineering stresses and stress tensor is a symmetric second rank tensor.

2.2. Other Invariants. Instead of Eqs. (2.1.5) other sets of invariants are possible [12, 28]. Each set of three linear-independent invariants is a unique set of invariants, so one can use the invariant sets equally. An important alternative set of invariants are the so-called irreducible invariants [12, 30, 31]:

$$\begin{aligned} I_1(\boldsymbol{\sigma}) &= \operatorname{tr}\boldsymbol{\sigma} = \boldsymbol{\sigma}\cdot\cdot\mathbf{I} \quad \text{linear invariant} \\ I_2(\boldsymbol{\sigma}) &= \boldsymbol{\sigma}\cdot\cdot\boldsymbol{\sigma} \quad \text{quadratic invariant} \\ I_3(\boldsymbol{\sigma}) &= (\boldsymbol{\sigma}\cdot\boldsymbol{\sigma})\cdot\cdot\boldsymbol{\sigma} \quad \text{cubic invariant} \end{aligned} \tag{2.2.1}$$

The conversion of the principal invariants to the irreducible invariants and vice versa are given in [12, 30], among others. Further invariant were introduced by Novozhilov [32, 33] or Lode [34, 35], among others.

Finally, let us present the principal invariants for the stress deviator

$$\mathbf{s} = \boldsymbol{\sigma} - \frac{1}{3}\operatorname{tr}\boldsymbol{\sigma}\mathbf{I} \tag{2.2.2}$$

Now we have

$$\begin{aligned} J_1(\mathbf{s}) &= \mathbf{s}\cdot\cdot\mathbf{I} = 0 \\ J_2(\mathbf{s}) &= -\frac{1}{2}J_1(\mathbf{s}^2) \\ J_3(\mathbf{s}) &= \det\mathbf{s} = \frac{1}{3}J_1(\mathbf{s}^3) \end{aligned} \tag{2.2.3}$$

3. Classical Criteria

3.1. The Very First Criteria. Possibly the most widespread equivalence hypothesis is the hypothesis of Huber [36, 37], von Mises [38] and Hencky [39]

$$\sigma_{\rm eq} = \sigma_{\rm vM} = \sqrt{\frac{3}{2}\mathbf{s}\cdot\cdot\mathbf{s}} \tag{3.1.1}$$

This condition was earlier formulated by J.C. Maxwell in a private letter to prospect Lord Kelvin [40]. One possible interpretation of the criterion (3.1.1) is that maximum distortion energy yields a limit state (yielding of a ductile material) which begins when the second invariant of the deviatoric stress $J_2(\mathbf{s})$ reaches a critical value [38]. This critical value can be identified experimentally, for example, in the tension test ($\sigma_{\rm Y}$) or torsion test ($\tau_{\rm Y}$). The critical shear stress τ_Y is related to the critical tension stress $\sigma_{\rm Y}$ by

$$\tau_{\rm Y} = \frac{\sigma_{\rm Y}}{\sqrt{3}} \tag{3.1.2}$$

It is obvious that this criterion does not consider the whole stress state expressed by the stress tensor. One can show that the first principal invariant is ignored which means compressibility of the material is excluded.

Possibly the oldest criterion is the maximum principal stress criterion. If $\sigma_{\rm I}$ is the maximum principal stress a critical state occurs if [41]

$$\sigma_{\rm I} \le \sigma_{\rm T} \tag{3.1.3}$$

Here $\sigma_{\rm T}$ is the positive tensile limit stress when failure occurs. The hypothesis is applied to materials which fail with separation fracture, without yielding:

- brittle materials (e.g. gray cast iron or welds) under predominantly static tensile loading and
- brittle and ductile materials under impact loading.

Note that the criterion is valid only in the case of maximum principal tension stresses. Otherwise the criteria contradicts Kachanov's statement concerning failure under compression [42]. It is obvious that also this criterium does not consider the whole stress state.

Henry Tresca [43] introduced a criterium assuming that the maximum shear stress is responsible for failure

$$\sigma_{eq} = 2\tau_{max} \tag{3.1.4}$$

With the maximum principal stresses the equivalent stress can be expressed as

$$\sigma_{eq} = \max\left(|\sigma_I - \sigma_{II}|, |\sigma_{II} - \sigma_{III}|, |\sigma_{III} - \sigma_I|\right) \tag{3.1.5}$$

The hypothesis is used for ductile materials with sliding fracture.

3.2. Criteria with One and More Parameters. A simple form of systematization of limit state criteria is according to the number of parameters to be determined experimentally for the limit state. However, it should be noted that there may be restrictions on the experiments that can be performed. If one takes the basic tests of materials testing [44], there is only a limited number of tests that produce homogeneous stress or distortion states in the material sample. In the following, we will again restrict ourselves to isotropic material behavior.

3.2.1. One-Parameter Criteria. The so-called one-parameter criteria are particularly simple, since only one test needs to be used.

- Maximum distortion criterion (Maxwell, Huber, von Mises, Hencky)

$$\sigma_{eq} = \sigma_{vM} = \sqrt{\frac{3}{2}\mathbf{s}\cdot\cdot\mathbf{s}} \le \sigma_u \tag{3.2.1}$$

Here is σ_u is the ultimate tensile stress when the limit case occurs. Equation (3.2.1) was presented earlier in this paper as (3.1.1).

- Maximum shear stress criterion (Coulomb, Tresca, St. Venant)

$$\tau_{max} = \frac{1}{2}(\sigma_I - \sigma_{III}) \le \frac{1}{2}\sigma_u \tag{3.2.2}$$

or

$$\sigma_{eq} = (\sigma_I - \sigma_{III}) \le \sigma_u \tag{3.2.3}$$

This criterion was introduced earlier in this paper by Eq. (3.1.4).

- Maximum principal strain criterion (Lamé, Clebsch, Rankine [41])

$$\sigma_{eq} = \sigma_I - \frac{1}{2}(\sigma_I - \sigma_{III}) \le \sigma_u \tag{3.2.4}$$

- Maximum principal stress criterion (Galilei, Leibniz)

$$\sigma_{eq} = \sigma_I \le \sigma_u \tag{3.2.5}$$

Note that the criterion is applicable only if $\sigma_I > 0$. In the previous section this criterion was given by Eq. (3.1.3).

- Maximum linear deformation criterion (St. Venant, Bach)

Let us assume the maximal linear strain ε_I as

$$\varepsilon_I = \sigma_I - \nu(\sigma_{II} + \sigma_{III}) \tag{3.2.6}$$

The criterion is expressed by using the Hooke's law. Finally, we have

$$\sigma_{eq} = \sigma_I - \nu(\sigma_{II} + \sigma_{III}) \le \sigma_u \tag{3.2.7}$$

Note that the last expression contains two parameters (ν and σ_u), but only one is related to the limit state (σ_u).

- Sdobyrev criterion [45, 46]

$$\sigma_{eq} = \frac{1}{2}(\sigma_{vM} + \sigma_I) \le \sigma_u \tag{3.2.8}$$

The Sdobyrev criterion is an averaging of the maximum distortion criterion and the maximum principal stress criterion. The limitations concerning the maximum principal stress are not presented in the original paper of Sdobyrev.

3.2.2. Two-Parameter Criteria. Now we are looking on two-parameter criteria based on two independent limit state tests.

- Mohr-Coulomb criterion [47–50]

$$\sigma_{eq} = \sigma_I - \xi\sigma_{III} \le \sigma_u \tag{3.2.9}$$

This criterion is a modification of maximum shear stress criterion (3.2.2). The additional parameter ξ is related to the ratio σ_u/σ_c. That means the compression test should be performed and the σ_c as the limit state stress at compression should be estimated. The Mohr–Coulomb criterion describes the response of brittle materials such as concrete, or rubble piles, to shear stress as well as normal stress. The theory applies to materials for which the compressive limit stress far exceeds the tensile limit stress.

- Botkin–Mirolyubov or Drucker–Prager criterion [51–54]

$$\sigma_{eq} = \sigma_I - \chi\sigma_{III} \le \sigma_u \tag{3.2.10}$$

- Pisarenko–Lebedev criterion [26, 55]

$$\sigma_{eq} = \frac{1}{2}[(1+\chi)\sigma_{vM} + (1-\chi)I_1] \le \sigma_u \tag{3.2.11}$$

This criterion can describe the von Mises equivalent stress and the influence of the hydrostatic stress state. Similar expression were given by Schleicher [56], Klebowski [57] and Nadai [58].

- Sandel criterion [59–61]

$$\sigma_{eq} = \sigma_I - \frac{1}{2}(1-\chi)\sigma_{II} - \chi\sigma_{III} \le \sigma_u \tag{3.2.12}$$

The Sandel criterion takes into account all principal stresses instead of the maximum principal stress (3.2.5) or the first and the third principal stress.

- Koval'chuk criterion [62]

$$\sigma_{eq} = \zeta\sigma_{vM} - (1-\zeta)(\sigma_I - \sigma_{III}) \le \sigma_u \tag{3.2.13}$$

The additional parameter ζ is defined as $\zeta = 2 - \frac{\varphi}{2} - \sqrt{3}$ with $\varphi = \frac{\sigma_u}{\tau_t}$ τ_t is the limit stress under torsion. This criterion is a combination of the von Mises criterion and the maximum shear stress criterion.

3.2.3. Three-Parameter and Four-Parameter Criteria. There are few three and four parameters criteria. The reason for this is the significantly increasing experimental effort. Let us introduce the following two criteria:

- Paul criterion [25, 63]

$$\sigma_{eq} = a_1\sigma_I + a_2\sigma_{II} + a_3\sigma_{III} \le \sigma_u \tag{3.2.14}$$

Here the three principal stresses are included independently.

- Birger criterion [64]

$$\sigma_{eq} = a_1\sigma_I + a_2\sigma_{II} + a_3\sigma_{III} + a_4\sigma_{vM} \le \sigma_u \tag{3.2.15}$$

Here the three principal stresses and the von Mises stress are included independently.

3.3. Final Comments. There are suggested much more simple criteria in the literature [7, 65–74], among others. However, it is difficult overall to determine the parameters experimentally. This has led to the use of the simplest possible criterion, where experiments can be easily planned and carried out. The possibilities of extending the classical criteria have been extensively discussed in recent years, with comprehensive studies on the development of new criteria cited in [13–15, 75–80], among others. However, with the increasing number of parameters and necessary tests the uniqueness of parameter identification is under question.

4. One Possible Classification

4.1. Generalized Formulation. Possible classifications of generalized limit criteria are widely discussed in the literature [12, 81] among others. Let us introduce here one simple classification including limit state criteria up to 6 parameters presented in [82]. The starting point is

$$\sigma_{\mathrm{eq}} \le \sigma_{\mathrm{u}}, \tag{4.1.1}$$

with σ_{eq} as a function of the stress tensor $\boldsymbol{\sigma}$. Assuming isotropic material behavior, the function $\sigma_{\mathrm{eq}} = \sigma_{\mathrm{eq}}(\boldsymbol{\sigma})$ can be simplified since in this particular case σ_{eq} is a function of three linear independent invariants. Any set of invariants can be used – all are equal. Here we introduce the following invariants

$$I_1 = \boldsymbol{\sigma}\cdot\cdot\mathbf{I}, \quad \sigma_{\mathrm{vM}} = \sqrt{\frac{3}{2}\mathbf{s}\cdot\cdot\mathbf{s}}, \quad \sin\xi = -\frac{9}{2}\frac{(\mathbf{s}\cdot\mathbf{s})\cdot\cdot\mathbf{s}}{\sigma_{\mathrm{vM}}^3} \quad \text{with} \quad \xi \le \frac{\pi}{6} \tag{4.1.2}$$

The last invariant is similar to the Lode angle [34, 35] and discussed in [32, 33]. The equivalent stress now is defined as

$$\sigma_{\mathrm{eq}} = \lambda_1\sigma_{\mathrm{vM}}\sin\xi + \lambda_2\sigma_{\mathrm{vM}}\cos\xi + \lambda_3\sigma_{\mathrm{vM}} + \lambda_4 I_1 + \lambda_5 I_1\sin\xi + \lambda_6 I_1\cos\xi \tag{4.1.3}$$

where λ_i $(i = 1,\dots,6)$ are scalar parameters which should be determined by experiments. Possible experiments are uniaxial tension with the limit stress σ_{u}, uniaxial compression with the limit stress σ_{c}, pure torsion with the limit stress τ_{u}, thin-walled tubular specimen under inner pressure, biaxial tension and uniaxial tension under superposed hydrostatic pressure. Only the first three test belongs to the basic tests in mechanical testing of materials [44] realizing homogeneous states. The remaining three tests can be exchanged at will.

4.2. Example. It can be shown that all previous discussed criteria can be expressed by (4.1.3). In the case of the Birger criterion this is shown in [83]. To determine the conditions of macroscopic fracture in a complex stress state, the critical (limit) stress $\sigma_{\mathrm{limit}}(T)$ must be compared with the equivalent stress σ_{eq}. T is the temperature. Let us consider that the temperature is constant. The further presentation of the problem of fracture criteria for a complex stress state will be carried out based on the ideas developed [64], where the value of σ_{eq} is defined as a function of principal stresses σ_i, $i = \mathrm{I,II,III}$, material parameters λ_j, $j = 0,1,2,3$ and internal state variables β_k, $k = 1,2,\dots$, reflecting the loading history and other effects [13, 84]. Rejecting the consideration of internal state variables (no influence of β_k) and accepting the linear dependence $\sigma_{\mathrm{eq}} = f(\sigma_1,\sigma_2,\sigma_3)$ we have a four-parameter fracture criterion (in a similar way a six-parameter criterion was introduced in [85])

$$\sigma_{\mathrm{eq}} = \lambda_0\sigma_{\mathrm{vM}} + \lambda_1\sigma_1 + \lambda_2\sigma_2 + \lambda_3\sigma_3 \le \sigma_{\mathrm{limit}} \equiv \sigma_{\mathrm{u}} \tag{4.2.1}$$

Table 1

Criterion	Expression	$\dfrac{\sigma_{\text{tension}}}{\sigma_{\text{compression}}}$	$\dfrac{\sigma_{\text{tension}}}{\sigma_{\text{torsion}}}$	$\dfrac{\sigma_{\text{tension}}}{\sigma_{\text{inner pressure}}}$
1	$\sigma_{\text{tension}} = \sigma_{\text{vM}}$	1	$\sqrt{3}$	$\dfrac{\sqrt{3}}{2}$
2	$\sigma_{\text{tension}} = \sigma_1 - \sigma_3 = 2\tau_{\text{torsion}}$	1	2	1
3	$\sigma_{\text{tension}} = \sigma_1$	0	1	1
4	$\sigma_{\text{tension}} = \sigma_1 - \frac{1}{2}(\sigma_2 + \sigma_3)$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{4}$
5	$\sigma_{\text{tension}} = \frac{1}{2}(\sigma_{\text{vM}} + \sigma_1)$	$\frac{1}{2}$	$\frac{1}{2}(\sqrt{3}+1)$	$\frac{1}{2}(\sqrt{3}+2)$
6	$\sigma_{\text{tension}} = \sigma_1 - \chi\sigma_3$	χ	$1+\chi$	1
7	$\sigma_{\text{tension}} = \frac{1}{2}(1+\chi)\sigma_{\text{vM}} + \frac{1}{2}(1-\chi)(\sigma_1+\sigma_2+\sigma_3)$	χ	$\frac{\sqrt{3}}{2}(1+\chi)$	$\frac{1}{4}\left(\sqrt{3}(1+\chi)+3(1-\chi)\right)$
8	$\sigma_{\text{tension}} = \chi\sigma_{\text{vM}} + (1-\chi)\sigma_1$	χ	$1+(\sqrt{3}-1)-\chi$	$1-\left(1-\frac{\sqrt{3}}{2}\right)\chi$

1 – Huber, von Mises, Hencky; 2 – Coulomb, Tresca, St. Venant;
3 – maxiumum principal stress (the stress must be positive), Galilei, Leibniz;
4 – maximum principal strain (special case if $\nu = 0.5$), Rankine, Lamé, Clebsch;
5 – Sdobyrev; 6 – Mohr;
7 – Nadai, Botkin–Mirolyubov, Schleicher, Prager–Drucker;
8 – Pisarenko–Lebedev

To define λ_0, λ_1, λ_2, λ_3 four basic experiments[2] are required. Using experiments on tension, compression, torsion and testing of thin-walled tube under inner pressure as such experiments, we obtain

$$\lambda_0 = \frac{1}{2-\sqrt{3}}\left(1 + \frac{\sigma_{\text{tension}}}{\sigma_{\text{compression}}} - \frac{\sigma_{\text{tension}}}{\tau_{\text{torsion}}}\right)$$
$$\lambda_1 = 1 - \lambda_0, \quad \lambda_2 = 2\frac{\sigma_{\text{tension}}}{\sigma_{\text{inner pressure}}} - 1 + \frac{\sigma_{\text{tension}}}{\sigma_{\text{compression}}} - \frac{\sigma_{\text{tension}}}{\tau_{\text{torsion}}} \tag{4.2.2}$$
$$\lambda_3 = \frac{1}{2-\sqrt{3}}\left[1 + (\sqrt{3}+1)\frac{\sigma_{\text{tension}}}{\sigma_{\text{compression}}} - \frac{\sigma_{\text{tension}}}{\tau_{\text{torsion}}}\right],$$

where σ_{tension}, $\sigma_{\text{compression}}$, τ_{torsion} and $\sigma_{\text{inner pressure}}$ are ultimate strengths (maximum of the stress-strain diagrams) for the listed types of loading. If we accept the values of the relations $\sigma_{\text{tension}}/\sigma_{\text{compression}}$, $\sigma_{\text{tension}}/\tau_{\text{torsion}}$ and $\sigma_{\text{tension}}/\sigma_{\text{inner pressure}}$ constant, then instead of (4.2.2) we have one-parameter criteria[3].

[2] The term "basic experiment" is under discussion. In solid mechanics basic experiments are tests, where homogeneous stress states are realized. Considering this and statements in material testing, such tests are [44]: tension, compression, torsion and hydrostatic compression.

[3] Other possible sets of relations are widely discussed in [12, 13, 15]. The question of the number of parameters for the fracture criteria is debatable and is determined by the uniqueness of the used material characteristics, which have sometimes a great scattering.

Using as basic two experiments (for example, tension and compression or tension and torsion), from which we determine the ratios $\lambda = \sigma_{\text{tension}}/\sigma_{\text{compression}}$ or $\lambda = \sigma_{\text{tension}}/\tau_{\text{torsion}}$, we obtain two-parameter criteria. The most popular criteria are one- and two-parameter criteria. They are obtained from the ratio (4.2.1) at various values of the ratio of strength characteristics, Table 1. Another example is given for the assumed 6-parameter criterion in [81].

Conclusions. The formulation of criteria will not lose its importance in the future, since mechanical engineering designs, but also in other applications, require simple engineering concepts for the evaluation of limit states. In contrast to the balance equations of continuum mechanics, which can be rationally justified, the formulation of limit criteria depends on many factors. Material or material, load, but also other factors play a role. Knowledge of the microstructure and its evolution explains some limit cases better, but for a macroscopic analysis one would quickly reach calculation times that are not realistic for industrial practice. Thus, there is also a great need for future research work on the limiting criteria. It would be important to expand the data basis so that more complex models can also find their way into practice.

Further research is also needed to overcome the limits of the isotropic model. The focus should be on two directions:

• Extension in the direction of anisotropic boundary criteria

Here, in particular, the question of mastering the rapidly increasing number of parameters needs to be addressed.

• Cyclic processes

Structures are often subject to cyclic stresses. Consequently, it has to be clarified here whether the equivalent stress expressions can be transferred to this case.

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On Some Limit State Criteria in Solid Mechanics

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Abstract—Limit states of the material behavior can be different. In the literature limit states are in particular defined as the transition from the elastic to the inelastic range (e.g. yield point), but also the loss of strength. Stress-, strain-, but also energy- and power-based concepts are used to describe the limit states. Dominant until today are the stress-based concepts. One of the most important issues here is the formulation of equivalence hypotheses that allow the material parameters, which are themselves scalar quantities, to be compared with corresponding equivalent quantities based on the stress tensor. In the paper are presented some equivalence hypotheses, a possible classification and open questions.

Keywords: limit state criteria, equivalent stress, basic tests in material testing