ЖЭТФ, 2022, том 162, вып. 5 (11), стр. 673–679

GRAVITATIONAL FIELD EFFECTS PRODUCED BY TOPOLOGICALLY NON-TRIVIAL GEOMETRY AND ROTATING FRAMES SUBJECT TO A COULOMB-TYPE SCALAR POTENTIAL

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Received July 17, 2022
revised version August 26, 2022,
Accepted for publication August 26, 2022

DOI: 10.31857/S0044451022110074
EDN: KYWSPL

1. INTRODUCTION

Rotation and rotating frames have always been a source of confusion while dealing with the problem of a uniformly rotating disk and its spatial geometry in the context of special theory of relativity (STR) [1]. An interesting feature in treating a rotational phenomena is the Galilean rotational transformation (GRT) between inertial (laboratory) frames and non-inertial rotating frames.

This coordinate transformation $\{x^\mu\} \to \{x'^\mu\}$ is defined by $(t \to t', r \to r', \phi \to \phi' + \Omega\, t', z \to z')$ [2–4], where Ω is the uniform angular speed of the rotating frame measured by an observer in the inertial frame. They had showed that the axial coordinate is restricted by $0 \le r < \frac{c}{\Omega}$ and others are usual ranges. Rotating frame of reference for various physical systems have been investigated in literature, for instance, on free scalar fields [5], on the Dirac particle [6], on a neutral particle [7], with quantum states under an electromagnetic field [8],

on the Dirac oscillator [9–11], on the Dirac particle subject to a hard-wall confining potential [12], on massive scalar fields [13], on spin-1 particles [14], on quantum fermionic fields inside a cylinder [15], on scalar bosons subject to Coulomb-type potential [16], on scattering problem of a non-relativistic particle [17], on spin-zero scalar particles in a space-time with space-like and spiral dislocations [18], on spin-zero scalar massive charged particles subject to Coulomb-type scalar and vector potentials [19], on spin-1/2 particles with a field and mixed potential [20], on the Casimir energy in a space-time with one extra compactified dimension [21], on spin-zero scalar particles in a space-time with magnetic screw dislocation [22], on the Dirac particles in an accelerated reference frame [23], on the Dirac fields in a space-time with spiral dislocation [24], on spin-zero scalar particles in a space-time with distortion of a vertical line to a vertical spiral [25], on the Klein-Gordon oscillator in a topologically non-trivial space-time [26] and in a cosmic string space-time with space-like dislocation [27], on spin-zero scalar particles in a Lorentz symmetry violation environment [28], on spin-zero scalar particles induced by the topology associated with a time-like dislocation space-time [29], on spin-zero scalar massive charged particles subject to Coulomb-type potential [30], on scalar particles [31,32], and the Klein–Gordon oscillator with scalar potential [33] in the context of Kaluza–Klein theory.

We are mainly interest on a space-time that is produced by a non-trivial topology defined by the geometry $\mathbf{S}^1 \times \mathbf{R}^3$, where $\mathbf{R}^3$ represents usual directions and $\mathbf{S}^1$ is a compact dimension (see fig. 1). The metric in polar coordinates (t', r', ϕ', θ') for this topologically non-trivial geometry is given by $ds^2 = -dt'^2 + dr'^2 + r'^2\, d\phi'^2 + R^2\, d\theta'^2$ [26].

For $\mathbf{S}^1$ rotating frame of reference, we perform the coordinate transformation from inertial frame (t, r, ϕ, θ) to the rotating frame $(t' = t, r' = r, \phi' = \phi, \theta' = \theta + \Omega\, t)$, one will have

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$$ds^2 = -\left(1 - R^2\,\Omega^2\right) dt^2 + dr^2 + r^2\,d\phi^2 + R^2\,d\theta^2 + 2\,\Omega\,R^2\,dt\,d\theta. \tag{1}$$

The ranges of the coordinate $0 < \theta < 2\,\pi$ and others are in the usual ranges. Here R is radius of the compact dimension $\mathbf{S}^1$, and the determinant of the corresponding metric tensor $g_{\mu\nu}$ is $det\,g = -r^2\,R^2$. An interesting feature one can see in contrast to the rotating Minkowski space-time is that the radius of the compact dimension $\mathbf{S}^1$ satisfies the condition $R < \frac{1}{\Omega}$ [26] such that the metric component g_{tt} is always negative otherwise this rotating system is physically unacceptable for $R > \frac{1}{\Omega}$.

2. GRAVITATIONAL FIELD EFFECTS UNDER ROTATING FRAME ON SCALAR BOSONS SUBJECT TO COULOMB-TYPE POTENTIAL

In this section, we study the relativistic quantum motions of scalar bosons subject to a Coulomb-type scalar potential in a topologically non-trivial rotating space-time. There are two ways that one can introduce a potential into the KG-equation. First one being an electromagnetic four-vector potential A_μ that can be introduced through a minimal substitution in momentum four-vector via $p_\mu \to (p_\mu - e\,A_\mu)$ or in the partial derivative via $\partial_\mu \to (\partial_\mu - i\,e\,A_\mu)$ [39], where e is the electric charges. This procedure has been widely used by several authors in literature [16,19,27,30–33,40–44]. The second procedure is to introduce a scalar potential $S(t,r)$ by modifying the mass term in the KG-equation via transformation $M^2 \to (M + S(t,r))^2$. This procedure has also been used by several authors to study the effects of potential in quantum systems [16,19,27,30–33,39–42].

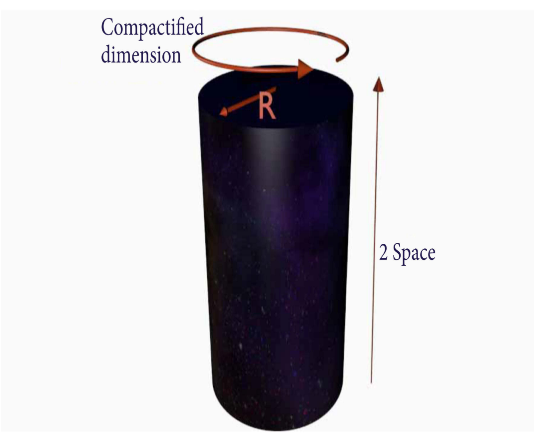

Fig. 1. Representation of the topologically non-trivial geometry $\mathbf{S^1} \times \mathbf{R^3}$ [26]

Thus, the quantum dynamics of scalar bosons subject to a potential $S(r)$ following the first approach is described by the wave equation [19,21–23,30–33,39–44]

$$\left[-\frac{1}{\sqrt{-g}}\,D_\mu\left(\sqrt{-g}\,g^{\mu\nu}\,D_\nu\right) + \left(M + S(r)\right)^2\right]\Psi = 0, \tag{2}$$

where M is the rest mass of the scalar bosons.

In this analysis, we have chosen the electromagnetic four-vector potential $A_\mu = (0, \vec{A})$ [22,27,33,42,44] with the following components

$$A_r = 0 = A_\theta \quad , \quad A_\phi = \frac{\Phi_B}{2\,\pi}, \tag{3}$$

where $\Phi_B = \Phi\,\Phi_0$ is the Aharnov-Bohm flux which is a constant, $\Phi_0 = \frac{2\,\pi}{e}$ is the amount of quantum flux, and Φ is the magnetic flux which is a positive integer. The presence of a magnetic flux in quantum system shows an analogue of the Aharonov-Bohm effect [37,38] which is a quantum mechanical phenomena that has been studied by many researchers in literature [27,30–33,41–44].

The Klein-Gordon equation (2) using (3) in the rotating space-time background (1) becomes

$$\left[-\left(\frac{\partial}{\partial\,t} - \Omega\,\frac{\partial}{\partial\,\theta}\right)^2 + \frac{1}{r}\,\frac{\partial}{\partial\,r}\left(r\,\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial}{\partial\phi} - i\,\Phi\right)^2 + \frac{1}{R^2}\,\frac{\partial^2}{\partial\,\theta^2}\right]\Psi = \left(M + S(r)\right)^2\Psi. \tag{4}$$

Several authors have been studied quantum motions of scalar and spin-half particles using potential of different kinds, such as the Cornell-type potential [40,41]. In this analysis, we are interested on another kind of potential proportional to the inverse of the axial distance. This type of potential is used for short-range interactions and called the Coulomb-type potential given by

$$S(r) \propto \frac{1}{r} \Rightarrow S(r) = \frac{\eta}{r}, \tag{5}$$

where $\eta > 0$ is a constant characterizes the potential parameter. This Coulomb-type potential has widely been studied in literature [41,43,45–57].

The total wave function $\Psi(t,r,\phi,z)$ can express in terms of a radial wave function $\psi(r)$ as follows:

$$\Psi(t,r,\phi,\theta) = e^{i\,(-E\,t + l\,\phi + q\,\theta)}\,\psi(r), \tag{6}$$

where E is energy of the scalar bosons, $l = 0, \pm1, \pm2, ..$ are the eigenvalues of the angular momentum operator $-i\,\hat{\partial}_\phi$, and q is a constant associated with the operator $-i\,\hat{\partial}_\theta$. Noted that for $\mathbf{S}^1$ compact dimension defined by a finite radius R satisfying the condition $R < \frac{1}{\Omega}$, the total wave function obeys the following condition

$$\Psi(\theta + 2\,\pi\,R) = \Psi(\theta). \tag{7}$$

Thereby, substituting the scalar potential (5) and the total wave function Eq. (6) into the Eq. (4), we have obtained the following radial wave equation

$$\psi''(r) + \frac{1}{r}\,\psi'(r) + \left[-\delta^2 - \frac{j^2}{r^2} - \frac{2\,\gamma}{r}\right]\psi(r) = 0, \tag{8}$$

where

$$\delta = \sqrt{M^2 + \frac{n^2}{R^2} - (E + \Omega\,n)^2}, \quad j = \sqrt{(l-\Phi)^2 + \eta^2},$$
$$\gamma = M\,\eta. \tag{9}$$

Performing a change of variables via $\xi = 2\,\delta\,r$ into the Eq. (8), we have

$$\psi''(\xi) + \frac{1}{\xi}\,\psi'(\xi) + \left(-\frac{j^2}{\xi^2} - \frac{\gamma}{\delta}\,\frac{1}{\xi} - \frac{1}{4}\right)\psi(\xi) = 0. \tag{10}$$

Suppose, a possible solution for the Eq. (10) in terms of a function $F(\xi)$ as:

$$\psi(\xi) = \xi^j\,e^{-\frac{\xi}{2}}\,F(\xi). \tag{11}$$

Substituting this solution (11) into the Eq. (10), we have obtained the following second-order differential equation:

$$\xi\,F''(\xi) + \Big(1 + 2\,j - \xi\Big)\,F'(\xi) + \left(-j - \frac{\gamma}{\delta} - \frac{1}{2}\right)F(\xi) = 0. \tag{12}$$

Equation (12) is the well-known confluent hypergeometric equation form [58, 59]. As state in Refs. [16, 19, 22, 26, 43, 51, 56, 58, 59], the solution to the differential equation of the form (12) can be expressed in terms of a confluent hyper-geometric function $F(\xi) = {}_1F_1\Big(j + \frac{\gamma}{\delta} + \frac{1}{2}, 2\,j + 1; \xi\Big)$ which is well-behaved for $\xi \to \infty$. Then, in searching for the bound-state solutions of the wave equation, the function ${}_1F_1$ must be a finite degree polynomial in ξ of degree n, and the quantity $\Big(j + \frac{\gamma}{\delta} + \frac{1}{2}\Big) = -n$ [16, 19, 22, 26, 43, 51, 56, 58, 59], where $n = 0, 1, 2, ...$

After simplifying this condition $\Big(j + \frac{\gamma}{\delta} + \frac{1}{2}\Big) = -n$, one will have the following expression of the energy eigenvalues:

$$E_{n,l,q} = -\Omega\,q \pm$$
$$\pm\left[M^2 + \frac{q^2}{R^2} - \frac{\eta^2}{\left(n + \sqrt{(l-\Phi)^2 + \eta^2} + \frac{1}{2}\right)^2}\right]^{1/2}. \tag{13}$$

The radial wave function is given by

$$\psi_{n,l}(\xi) = \xi^{\sqrt{(l-\Phi)^2+\eta^2}}\,e^{-\frac{\xi}{2}}\times$$
$$\times\,{}_1F_1\left(j + \frac{\gamma}{\delta} + \frac{1}{2}, 2\,j + 1; \xi\right). \tag{14}$$

Equation (13) is the relativistic energy eigenvalue and Eq. (14) is the radial wave function of the scalar bosons in a topologically non-trivial rotating space-time subject to a Coulomb-type external potential. We can see that the eigenvalue solution is modified by the non-trivial topology of the geometry defined by the radius R, and the Coulomb-type potential. We also see that the energy levels are shifted by rotating frame of reference, and hence, these are not equally spaced on either side about $E_{n,l,q} = 0$ for constant values of l, q. This effect arises due to the coupling between the quantum number $q \neq 0$ and the uniform angular speed Ω of rotating frame of reference.

In Ref. [26], authors studied the Klein-Gordon oscillator in a non-trivial topological space-time geometry. They solved the wave equation analytically and obtained the following energy eigenvalue expression (see Eq. (28) there and we have replaced $n \to q$)

$$E_{\pm} = \pm\sqrt{M^2 + \frac{q^2}{R^2} + 2\,M\,\omega\,(2\,N' + |l|)}, \tag{15}$$

where $N' = N + 1 = 1, 2, 3, ...$

One can easily show that the presented energy eigenvalue (13) is completely different from the result (15) obtained in Ref. [26]. This is because, we have considered a non-inertial reference frame which rotates with constant angular speed Ω, the Coulomb-type scalar potential characterise by the parameter η as well as the magnetic flux Φ which shifts the energy levels and the wave function. Thus, our presented result in this section is completely new and different from the previous result given in Ref. [26].

3. GRAVITATIONAL FIELD EFFECTS UNDER ROTATING FRAME ON KG-OSCILLATOR SUBJECT TO COULOMB-TYPE SCALAR POTENTIAL

In this section, we will study the Klein-Gordon oscillator [60] subject to an external potential in a topologically non-trivial four-dimensional rotating space-time. In Ref. [26], authors studied the KG-oscillator in this topologically non-trivial rotating space-time without any external potential. In this work, we have inserted a Coulomb-type external potential and magnetic flux as stated earlier and analyze their effects on the eigenvalue solution of the oscillator fields. The KG-oscillator analogous to the Dirac oscillator [61] has attracted attention among researchers in current times (see, Refs. [19, 22, 26, 27, 33, 57, 62]). The KG-oscillator is examined by the replacements of the radial momentum vector [19, 22, 26, 27, 33, 57, 62]

$$\vec{p} \to (\vec{p} - i\,M\,\omega\,\vec{r}), \quad \vec{p}^{\dagger} \to (\vec{p} + i\,M\,\omega\,\vec{r}), \tag{16}$$

where ω is the frequency of the oscillator fields, and r being distance from the particle to the axis of symmetry.

Therefore, the Klein-Gordon oscillator equation is given by

$$\Big[-\frac{1}{\sqrt{-g}} \Big(D_\mu + M\,\omega\,X_\mu\Big) \times \times \Big\{ \sqrt{-g}\,g^{\mu\nu} \Big(D_\nu - M\,\omega\,X_\nu\Big) \Big\} + + \Big(M + S(r)\Big)^2 \Big]\,\Psi = 0, \tag{17}$$

where $X_\mu = (0, r, 0, 0) = r\,\delta^r_\mu$ is a four-vector.

Explicitly witting the KG-oscillator equation (17) in the rotating space-time background (1) and using the electromagnetic potential Eq. (3) and the external potential Eq. (5), we have

$$\Big[-\Big(\frac{\partial}{\partial t} - \Omega\,\frac{\partial}{\partial \theta}\Big)^2 + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\,\frac{\partial}{\partial r} - M^2\,\omega^2\,r^2 - 2\,M\,\omega + \frac{1}{r^2}\,\Big(\frac{\partial}{\partial \phi} - i\,\Phi\Big)^2 + \frac{1}{R^2}\,\frac{\partial^2}{\partial \theta^2} \Big]\,\Psi = \Big(M + \frac{\eta}{r}\Big)^2\,\Psi. \tag{18}$$

Substituting the wave function (6) into the Eq. (18), we have obtained the following radial wave equation:

$$\psi''(r) + \frac{1}{r}\,\psi'(r) + \Big[\Lambda - M^2\,\omega^2\,r^2 - \frac{j^2}{r^2} - \frac{2\,\gamma}{r}\Big]\,\psi(r) = 0, \tag{19}$$

where j, γ are defined in Eq. (9) and

$$\Lambda = (E + \Omega\,q)^2 - M^2 - 2\,M\,\omega - \Big(\frac{q}{R}\Big)^2. \tag{20}$$

Let us now perform a change of variables via $x = \sqrt{M\,\omega}\,r$. Then, Eq. (19) can be rewritten as

$$\psi''(x) + \frac{1}{x}\,\psi'(x) + \Big[\frac{\Lambda}{M\,\omega} - x^2 - \frac{\varsigma}{x} - \frac{j^2}{x^2}\Big]\,\psi(x) = 0, \tag{21}$$

where $\varsigma = \frac{2\,\gamma}{\sqrt{M\,\omega}}$.

As stated earlier the wave function $\psi(x)$ is well-behaved and regular everywhere. Suppose, a possible solution to the above radial wave equation Eq. (21) is given by

$$\psi(x) = x^j\,e^{-\frac{x^2}{2}}\,H(x), \tag{22}$$

where $H(x)$ is an unknown function.

Thereby, substituting the radial wave function Eq. (22) into the Eq. (21), we have

$$H''(x) + \Big[\frac{1 + 2\,j}{x} - 2\,x\Big]\,H'(x) + \Big[-\frac{\varsigma}{x} + \Xi\Big]\,H(x) = 0, \tag{23}$$

where $\Xi = \frac{\Lambda}{M\,\omega} - 2\,(1 + j)$.

Equation (23) is the biconfluent Heun differential equation form [22, 32, 33, 40, 42] and $H(x)$ is the Heun function. Substituting a power series expansion

$$H(x) = \sum_{i=0}^{\infty} d_i\,x^i$$

[59] into the Eq. (23), we have obtained few coefficients

$$d_1 = \Big(\frac{\varsigma}{1 + 2\,j}\Big)\,d_0, \quad d_2 = \frac{1}{4\,(1 + j)}\,\Big[\varsigma\,d_1 - \Xi\,d_0\Big]$$

with the following recurrence relation

$$d_{m+2} = \frac{1}{(m + 2)(m + 2 + 2\,j)}\,\Big[\varsigma\,d_{m+1} - (\Xi - 2\,m)\,d_m\Big]. \tag{24}$$

One can see this power series expansion $H(x)$ becomes a polynomial of finite degree m by imposing the following two conditions [22, 32, 33, 40, 42]

$$\Xi = 2\,m \quad (m = 1, 2, ...) \quad , \quad d_{m+1} = 0. \tag{25}$$

By analyzing the first condition, we have obtained following energy eigenvalue $E_{m,l,q}$ expression:

$$E_{m,l,q} = -\Omega\,q \pm \pm \Big[M^2 + 2\,M\,\omega_{m,l} \times \times \Big(m + \sqrt{(l - \Phi)^2 + \eta^2} + 2\Big) + \frac{q^2}{R^2} \Big]^{1/2}. \tag{26}$$

The corresponding radial wave function is given by

$$\psi_{m,l}(x) = x^{\sqrt{(l-\Phi)^2 + \eta^2}}\,e^{-\frac{x^2}{2}}\,H(x), \tag{27}$$

where $H(x)$ is now a finite degree polynomial of degree m.

Finding solutions of the quantum system still not complete because one must analyze the second condition $d_{m+1} = 0$ one by one to get the complete information of a quantum state. As example, for the radial mode $m = 1$, we have $\Xi = 2$ and $d_2 = 0$ which gives us a constraint on the oscillation frequency $\omega \to \omega_{1,l}$ given by

$$\omega_{1,l} = \left(\frac{M\,\eta^2}{\sqrt{(l-\Phi)^2+\eta^2} + \frac{1}{2}} \right). \tag{28}$$

Therefore, the ground state energy level associated with the radial mode $m = 1$ is given by

$$E_{1,l,q} = -\Omega\, q \pm \pm M \sqrt{1 + 2\,\eta^2 \left(\frac{\sqrt{(l-\Phi)^2+\eta^2} + 3}{\sqrt{(l-\Phi)^2+\eta^2} + \frac{1}{2}} \right) + \left(\frac{q}{M\,R}\right)^2}. \tag{29}$$

And the ground state radial wave function is given by

$$\psi_{1,l}(x) = x^{\sqrt{(l-\Phi)^2+\eta^2}}\, e^{-\frac{x^2}{2}} \times \times \left(1 + \frac{x}{\sqrt{\sqrt{(l-\Phi)^2+\eta^2} + \frac{1}{2}}} \right) d_0. \tag{30}$$

Similarly, for the radial mode $m = 2$, we have $\Xi = 4$ and $d_3 = 0$ which gives us another constraint on the oscillation frequency $\omega \to \omega_{2,l}$ given by

$$\omega_{2,l} = \frac{1}{2} \left(\frac{M\,\eta^2}{\sqrt{(l-\Phi)^2+\eta^2} + 1} \right), \tag{31}$$

Therefore, the first excited state energy level of the bound-states solution defined by the radial mode $m = 2$ is given by

$$E_{2,l,q} = -\Omega\, q \pm \pm M \sqrt{1 + \eta^2 \left(\frac{\sqrt{(l-\Phi)^2+\eta^2} + 3}{\sqrt{(l-\Phi)^2+\eta^2} + 1} \right) + \left(\frac{q}{M\,R}\right)^2}. \tag{32}$$

And the corresponding radial wave function is given by

$$\psi_{2,l}(x) = x^{\sqrt{(l-\Phi)^2+\eta^2}}\, e^{-\frac{x^2}{2}}\, (d_0 + d_1\, x + d_2\, x^2), \tag{33}$$

where

$$d_1 = 2 \left(\frac{\sqrt{\sqrt{(l-\Phi)^2+\eta^2} + \frac{3}{4}}}{\sqrt{(l-\Phi)^2+\eta^2} + \frac{1}{2}} \right) d_0,$$
$$d_2 = \left(\frac{1}{\sqrt{(l-\Phi)^2+\eta^2} + \frac{1}{2}} \right) d_0. \tag{34}$$

We can see that the energy eigenvalues and the wave function are modified by the non-trivial topology of the space-time geometry, and the Coulomb-type potential. One can show that the presented energy eigenvalue gets modified in comparison to those result obtained in [26] due to the presence of the Coulomb-type external potential and the magnetic quantum flux. This Coulomb-type external potential is responsible for the bound-state solutions, and thus, the ground state is defined by the radial quantum number $n = 1$ instead of $n = 0$.

4. CONCLUSIONS

In this analysis, we have determined solutions of the wave equation under the effects of the gravitational field produced a topologically non-trivial geometry subject to a Coulomb-type external potential in a rotating frame of reference. We have seen that the non-trivial topology of the geometry defined by the radius R of the compact dimension, and the Coulomb-type external potential modified the eigenvalue solutions. Furthermore, the presence of the magnetic flux causes a change in the angular quantum number $l \to l_0 = \left(l - \frac{e\,\Phi_B}{2\,\pi}\right)$ which shows that the energy eigenvalue depends on the geometric quantum phase. This dependence of the eigenvalue on the geometric quantum phase gives us the gravitational analogue to the Aharonov-Bohm effect [37, 38]. Several authors have been investigated this quantum mechanical effect in literature (*e. g.*, [27, 30, 31, 33]). Also, we have seen a coupling between the angular quantum number q and the uniform angular speed Ω of the rotating frame of reference. This coupling causes asymmetry in the relativistic energy levels, and hence, are not equally spaced on either side about $E_{n/m,l,q} = 0$ for constant values of l, q.

We has seen that the presence of Coulomb-type potential allowed the formation of bound-state solutions and causes difference in results with those obtained in Ref. [26]. Another point we have noticed is that the rotating frames restricted the radius of compact circle $\mathbf{S^1}$ in the range $R < \frac{1}{\Omega}$, and an analogous to the Sagnac-type effect [6, 10, 27, 33] is observed due to the coupling between the quantum number q and uniform angular speed Ω of rotating frames. This coupling causes asymmetry in the energy levels and therefore, they are not equally spaced on either side about $E_{n,l,q} = 0$ for constant values of l, q.

The full text of this paper is published in the English version of JETP.

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