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AN ANTISYMMETRIC SOLUTION OF THE 3D INCOMPRESSIBLE NAVIER–STOKES EQUATIONS WITH “TORNADO-LIKE” BEHAVIOR

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Abstract. We consider a solution of the incompressible Navier–Stokes equations in $\mathbb{R}^3$, related to the singular complex solutions of Li and Sinai [1], and such that a growth of the enstrophy $S(t)$ is expected. The computer simulations show that $S(t)$ increases up to a time T_E (singularities are excluded by axial symmetry). They also reveal an interesting “tornado-like” behavior, with a sharp increase of speed and vorticity in an annular region, as for some “extreme” weather phenomena.

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1. Introduction. In recent times, results obtained by variants of the Navier–Stokes (NS) system indicate that there are indeed smooth solutions that become singular (blow-up) at a finite time [2, 3]. As for evidence from computer simulations, the NS equations in 3D are in general difficult to follow for high values of the velocity and the vorticity, and in absence of reliable theoretical guide-lines they are inconclusive (see, e. g., [4]).

The singular solutions, if they exist, would describe sudden concentrations of energy in a finite region, as it happens in tornadoes or hurricanes, for which no effective model is now available. In fact, the main features of the possible finite-time singularities (“blow-up”), are the divergence of the total enstrophy [5], and the divergence at some point of the absolute value of the velocity [6].

The paper [1] is a first step in a plan to prove the existence of a blow-up. The main idea is to apply dynamical system techniques in order to control the transfer of energy to the fine scales. The approach can also be applied to other models [7]. We give here a brief description.

Consider the NS system in the whole space $\mathbb{R}^3$ with no forcing, and viscosity ν

$$\frac{\partial \mathbf{u}}{\partial t}(\mathbf{x},t) + \sum_{j=1}^{3} u_j(\mathbf{x},t)\frac{\partial \mathbf{u}}{\partial x_j}(\mathbf{x},t) = \\ = \nu\Delta\mathbf{u}(\mathbf{x},t) - \nabla p(\mathbf{x},t), \quad \mathbf{x}=(x_1,x_2,x_3)\in\mathbb{R}^3, \quad (1)$$

$$\nabla\cdot\mathbf{u}(\mathbf{x},t) = \sum_j \frac{\partial u_j}{\partial x_j}(\mathbf{x},t) = 0, \quad \mathbf{u}(\mathbf{x},0)=\mathbf{u}_0(\mathbf{x}), \quad (2)$$

where p is the pressure. Assuming $\nu = 1$ and passing to the modified Fourier transform

$$\mathbf{v}(\mathbf{k},t) = \frac{i}{(2\pi)^3}\int_{\mathbb{R}^3} \mathbf{u}(\mathbf{x},t)e^{-i\langle\mathbf{k},\mathbf{x}\rangle}d\mathbf{x}, \\ \mathbf{k}=(k_1,k_2,k_3)\in\mathbb{R}^3, \quad (3)$$

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where $\langle \cdot, \cdot \rangle$ is the scalar product, the equation (1) can be written as a single integral equation:

$$\mathbf{v}(\mathbf{k},t) = e^{-t\mathbf{k}^2}\mathbf{v}_0(\mathbf{k}) + \int_0^t e^{-(t-s)|\mathbf{k}|^2} \times \\ \times \int_{\mathbb{R}^3} \langle \mathbf{v}(k-k',s), \mathbf{k}\rangle \, P_{\mathbf{k}}\mathbf{v}(\mathbf{k}',s)\, d\mathbf{k}' ds, \quad (4)$$

where $P_{\mathbf{k}}\mathbf{v} = \mathbf{v} - \frac{\langle \mathbf{v}, \mathbf{k}\rangle}{|\mathbf{k}|^2}\mathbf{k}$ is the solenoidal projector and $\mathbf{v}_0$ the transform of $\mathbf{u}_0$. In general $\mathbf{v}(\mathbf{k},t)$ is a complex function. Li and Sinai consider real solutions of (4), which give in general complex solutions of (1), but if $\mathbf{v}_0(\mathbf{k})$ is antisymmetric, the solution $\mathbf{u}(\mathbf{x},t)$ is also real and antisymmetric in $\mathbf{x}$.

Multiplying $\mathbf{v}_0$ by a constant A, which controls the initial energy, the solution of (4) is written as

$$\mathbf{v}_A(\mathbf{k},t) = A\mathbf{g}^{(1)}(\mathbf{k},t) + \\ + \int_0^t e^{-\mathbf{k}^2(t-s)} \sum_{p=2}^{\infty} A^p \mathbf{g}^{(p)}(\mathbf{k},s) ds, \quad (5)$$

where

$$\mathbf{g}^{(1)}(\mathbf{k},s) = e^{-s\mathbf{k}^2}\mathbf{v}_0(\mathbf{k}),$$

$$\mathbf{g}^{(2)}(\mathbf{k},s) = \int_{\mathbb{R}^3} \left\langle \mathbf{g}^{(1)}(\mathbf{k}-\mathbf{k}',s), \mathbf{k}\right\rangle P_{\mathbf{k}}\mathbf{g}^{(1)}(\mathbf{k}',s)\, d\mathbf{k}'$$

and for $p > 2$

$$\mathbf{g}^{(p)}(\mathbf{k},s) = \sum_{\substack{p_1+p_2=p \\ p_1,p_2>1}} \int_0^s ds_1 \int_0^s ds_2 \times \\ \times \int_{\mathbb{R}^3} \left\langle \mathbf{g}^{(p_1)}(\mathbf{k}-\mathbf{k}',s_1), \mathbf{k}\right\rangle P_{\mathbf{k}}\mathbf{g}^{(p_2)}(\mathbf{k}',s_2) \times \\ \times e^{-(s-s_1)(\mathbf{k}-\mathbf{k}')^2-(s-s_2)(\mathbf{k}')^2} d\mathbf{k}' + \\ + \text{boundary terms.} \quad (6)$$

The boundary terms involve $\mathbf{g}^{(1)}$, and it can be shown that the series converges for small t.

If, as in [1], the support of $\mathbf{v}_0$ is concentrated in a sphere K_R of radius R centered around the point $\mathbf{k}^{(0)}$ with $|\mathbf{k}^{(0)}| \gg R$, then $\mathbf{g}^{(p)}$, which is a convolution, has a support centered around $p\mathbf{k}^{(0)}$ with an effective diameter of the order $\mathcal{O}(\sqrt{p})$. By a standard rescaling we write $\tilde{\mathbf{g}}^{(p)}(\mathbf{Y},s) = \mathbf{g}^{(p)}(p\mathbf{k}^{(0)} + \sqrt{p}\,\mathbf{Y},s)$, and consider for large p the map $\tilde{\mathbf{g}}^{(p)} \to \tilde{\mathbf{g}}^{(p+1)}$. The possible fixed points of that map control the excitation of the high $\mathbf{k}$-modes, i. e., of the fine structure components of $\mathbf{u}(\mathbf{x},t)$.

The following Ansatz is formulated in [1]: for a class of Gaussian dominated initial data with support as above, as $p \to \infty$ the following asymptotics holds

$$\tilde{\mathbf{g}}^{(p)}(\mathbf{Y},s) \sim p(\Lambda(s))^p \prod_{i=1}^{3} g(Y_i)(\mathbf{H}(\mathbf{Y}) + \\ + \delta^{(p)}(\mathbf{Y},s)), \quad \mathbf{Y} = (Y_1,Y_2,Y_3), \quad (7)$$

where $\mathbf{H}$ is a fixed point, $g(x) = \exp(-x^2/2)/\sqrt{2\pi}$, Λ is a strictly increasing smooth positive function and $\delta^{(p)}(\mathbf{Y},s) \to 0$ as $s \to \infty$. $\mathbf{H}$ is in fact a plane vector, as its component along $\mathbf{k}^{(0)}$ vanishes by incompressibility.

The Ansatz (7) is proved in [1] for $\mathbf{k}^{(0)} = (0,0,a)$, $a > 0$ and $\mathbf{H}^{(0)}(\mathbf{Y}) = c\,(Y_1,Y_2,0)$, with $c > 0$, and a finite-time blow-up is proved for a class of initial data $\mathbf{v}_0$. Both the total enstrophy and the total energy diverge as $t \uparrow \tau$ (for complex function the energy equality holds but it is not coercive).

The object of our paper is the real flow which follows from initial data obtained by antisymmetrizing the data associated to solutions that blow-up, namely

$$\mathbf{v}_0(\mathbf{k}) = \left(k_1, k_2, -\frac{k_1^2+k_2^2}{k_3}\right) g(k_1)g(k_2) \times \\ \times \left[g(k_3-a)\chi_b(|\mathbf{k}-\mathbf{k}^{(0)}|) + g(k_3+a)\chi_b(|\mathbf{k}+\mathbf{k}^{(0)}|)\right], \quad (8)$$

where $a > b \gg 1$ and $\chi_b(r)$ is smooth with $\chi_b(r) = 0$ if $r \geq b$, $\chi_b(r) = 1$ if $0 \leq r \leq b-\epsilon$, for ϵ small enough. The support of (8) is made of two regions around $\pm\mathbf{k}^{(0)}$, and the convolution $\mathbf{g}^{(p)}$ is a sum of terms centered around the points $(0,0,\ell a)$, $\ell = -p,\ldots,p$, with the main contribution coming for $|\ell| = \mathcal{O}(\sqrt{p})$. As the components $\mathbf{g}^{(p)}$ for large p are excited, the support moves quickly to the high $\mathbf{k}$ region, and an increase of the enstrophy is expected.

2. Results of computer simulations. We used a special program for solutions of the integral equation (4), created for the purpose of following the blow-up of the complex solutions, as described in [8], where complex solutions of (4) could be followed up to times close to the critical blow-up time. Our mesh in $\mathbf{k}$-space is a regular lattice centered at the origin with step $\delta = 1$, with maximal configuration $[-254,254] \times \times [-254,254] \times [-3000,3000]$. We deal with about $5\cdot 10^9$ real numbers, close to the maximal capacity of modern supercomputers.

Our main aim was to follow the behavior of the enstrophy and of the marginal distributions of the square vorticity in $\mathbf{k}$-space, which describe the flow of energy

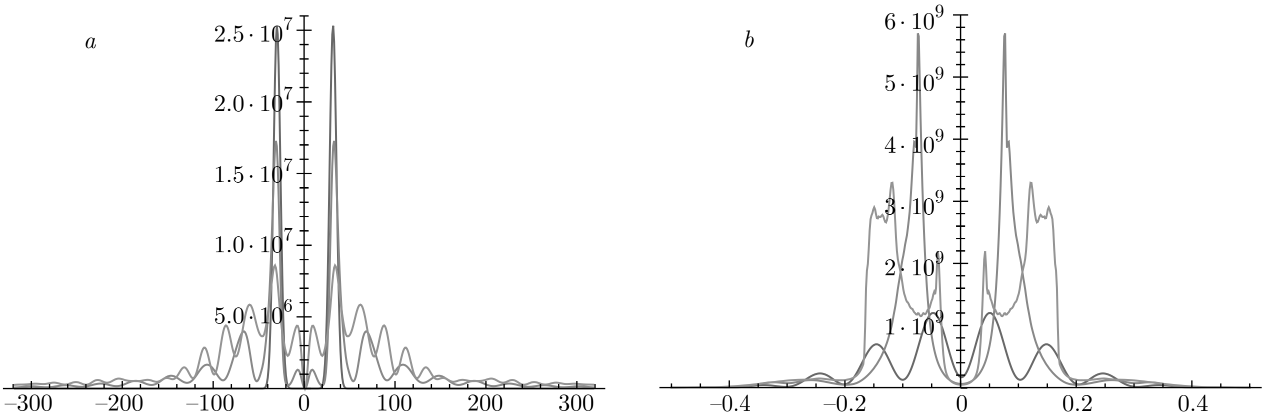

Fig. 1. (Color online) Plots of the marginal distributions $S_3(k_3,t)$ (a) and $\tilde{S}_3(x_3,t)$ (b) at the times $t=0$ (blue), $t=400\tau\approx T_V$ (red), and $t=711\tau\approx T_M$ (green)

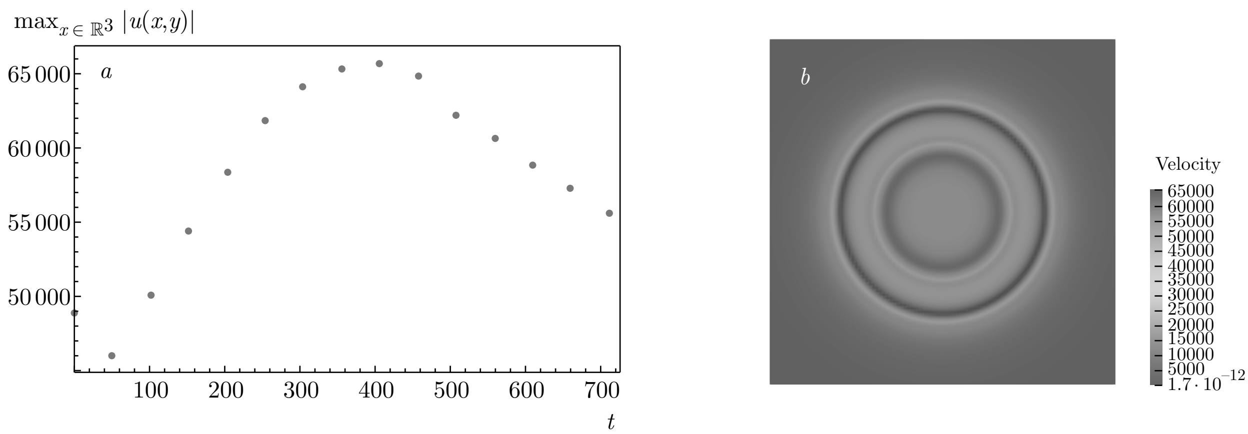

Fig. 2. (Color online) Plot of the maximal velocity as a function of time (a), and the absolute value of the velocity field $|\mathbf{u}(\mathbf{x},t)|$ on the plane $x_3=0.08$ at the time $t=450\tau$ (b)

to the microscale in physical space, and their behavior is stable with respect to refinements of the mesh. An analysis comparing the accuracy of our program with respect to that of finite-difference methods is under way.

We consider the initial data (8) with $a=30$ and multiplied by a constant A is such that the initial energy is $E_0=62\cdot10^6$. The study of the behavior of the solutions as the parameters a and A vary is under way. As for the complex case [8] the large initial data ensure a short running time, which makes simulations possible. In what follows time is measured in units of $\tau=1.5625\cdot10^{-8}$.

As expected, the total enstrophy $S(t) = \int_{\mathbb{R}^3}|\omega(\mathbf{x},t)|^2d\mathbf{x}$, where $\omega(\mathbf{x},t)$ is the vorticity field, increases sharply up to a critical time $T_E\approx711\tau$ and then decays. In Fig. 1 we report the evolution in time of the marginal densities of the enstrophy along the third axis: in **k**-space (a) $S_3(k_3,t)=\int_{\mathbb{R}^2}|\mathbf{k}|^2|\mathbf{v}(\mathbf{k},t)|^2dk_1dk_2$, and in **x**-space (b) $\tilde{S}_3(x_3,t)=\int_{\mathbb{R}^2}|\omega(\mathbf{x},t)|^2dx_1dx_2$. On Fig. 1$a$ we can see how the support moves into the high $|\mathbf{k}|$-region. As time grows the peaks of $S_3(k_3,t)$ tend to be close to the values $k_3\approx ja$, with $j=\pm1,\pm2,\ldots$ (green line), a modulated periodicity corresponding in the plot of $S_3(x_3)$ on Fig. 1b to two symmetric peaks at $x_3=\pm\bar{x}_3$ where $\bar{x}_3$ grows in time but and as $t\uparrow T_E$ approaches a value $\bar{x}_3\approx\pi/a$.

A remarkable fact is that the maximal value of the speed $|\mathbf{u}(\mathbf{x},t)|$ also grows, up to a time $T_V\approx400\tau$ (Fig. 2a). Moreover as we approach the time T_V the high values of $|\mathbf{u}(\mathbf{x},t)|$ are concentrated in an annular region around a plane $x_3=\bar{x}_3$ as shown by Fig. 2b. The maximal values of the vorticity also grow, up to time T_V and are concentrated in the same annular region.

3. Concluding remarks. The flow exhibits a “tornado-like” behavior, such as a rapid increase of the absolute values of velocity and vorticity in a confined region, indicating that similar flows could provide a model for some class of such “extreme” phenomena. (Similar solutions probably exist also for compressible fluids in a quasi-incompressible regime.)

A real blow-up is however excluded, due to the fact that the solutions obtained by antisymmetrizing the initial data which in [7] lead to the fixed point $\mathbf{H}^{(0)}$ are close to axial symmetry and, by a recent result [9], they cannot become singular.

We expect however that the results of [1] on the blow-up hold also for other fixed points $\mathbf{H} \neq \mathbf{H}^{(0)}$ which are not axial symmetric. The next steps in the proof of a negative solution of the GRP should be the extension of the Li–Sinai theory to such fixed points and a rigorous analysis of the real solutions related to them. In absence of theoretical results, important information can be obtained by computer simulations, which can also reveal physically relevant details.

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