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Gauge equivalence between 1 + 1 rational Calogero–Moser field theory and higher rank Landau–Lifshitz equation

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The $1+1$ field generalization of the Calogero–Moser model was proposed in [1, 2], see also [3]. The Hamiltonian is given by the following expression:

$$\mathcal{H}^{\rm 2dCM} = \oint \mathrm{d}x\, H^{\rm 2dCM}(x),$$

$$H^{\rm 2dCM}(x) = \sum_{i=1}^{N} p_i^2 \left(c - kq_{ix}\right) - \frac{1}{Nc}\left(\sum_{i=1}^{N} p_i \left(c - kq_{ix}\right)\right)^2 - \sum_{i=1}^{N} \frac{k^4 q_{ixx}^2}{4\left(c - kq_{ix}\right)} + \frac{k^3}{2}\sum_{i\neq j}^{N} \frac{q_{ix}q_{jxx} - q_{jx}q_{ixx}}{q_i - q_j} - \frac{1}{2}\sum_{i\neq j}^{N} \frac{1}{\left(q_i - q_j\right)^2}\Big[\left(c - kq_{ix}\right)^2\left(c - kq_{jx}\right) + \left(c - kq_{ix}\right)\left(c - kq_{jx}\right)^2 - ck^2\left(q_{ix} - q_{jx}\right)^2\Big], \qquad (1)$$

where x is the (space) field variable and $k \in \mathbb{C}$ is a constant parameter. The momenta p_i and coordinates q_j are canonically conjugated fields: $\{q_i(x), p_j(y)\} = \delta_{ij}\delta(x-y)$. The model (1) is integrable in the sense that it has algebro-geometric solutions and equations of motion are represented in the Zakharov–Shabat (or Lax or zero curvature) form: $\partial_t U(z) - k\partial_x V(z) + [U(z), V(z)] = 0$, where U-V pair $U^{\rm 2dCM}(z), V^{\rm 2dCM}(z) \in \mathrm{Mat}(N, \mathbb{C})$ is a pair of matrix valued functions of the fields $p_j(x)$, $q_j(x)$, $j = 1, ..., N$ and their derivatives. They also depend on the spectral parameter z. Explicit expression for U-V pair can be found in [1, 2]. It was argued in [3] that there exist a gauge transformation $G(z) \in \mathrm{Mat}(N, \mathbb{C})$, which transforms U-V pair for the field Calogero–Moser model to the one for some Landau–Lifshitz type model:

$$U^{\rm LL}(z) = G(z)U^{\rm 2dCM}(z)G^{-1}(z) + k\partial_x G(z)G^{-1}(z). \qquad (2)$$

For the $N = 2$ case explicit construction of the matrix $G(z)$ and the change of variables was derived in our paper [4], and the Landau–Lifshitz model for GL_2 rational R-matrix was derived in [5]. The goal of this article is to define the gauge transformation in gl_N case, describe the corresponding Landau–Lifshitz type model and find explicit change of variables using relation (2).

Recently the $1+1$ field generalization of the so-called rational top model was suggested in [6]. It is given by Landau–Lifshitz type equation, i.e. the field variables are arranged into $N \times N$ matrix S and the Poisson structure is linear: $\{S_{ij}(x), S_{kl}(y)\} = N^{-1}\Big(S_{il}(x)\delta_{kj} - S_{kj}(x)\delta_{il}\Big)\delta(x-y)$. The construction of the Landau–Lifshitz equation and its U-V pair is based on R-matrix satisfying the associative Yang–Baxter equation [7, 8]: $R^\hbar_{12}R^\eta_{23} = R^\eta_{13}R^{\hbar-\eta}_{12} + R^{\eta-\hbar}_{23}R^\hbar_{13}$, $R^x_{ab} = R^x_{ab}(z_a - z_b)$. Suppose $\mathrm{rank}(S) = 1$, so that $S^2 = cS$, $c = \mathrm{tr}(S)$. Then the Landau–Lifshitz equation reads:

$$\partial_t S = k^{-2}c\,[S, \partial_x^2 S] + 2c\,[S, J(S)] - 2k[S, E(\partial_x S)]\,, \qquad (3)$$

where $E(S) = \mathrm{tr}_2\Big(r^{(0)}_{12}\ \overset{2}{S}\Big)$, $\overset{2}{S} = 1_N \otimes S$ and $J(S) = \mathrm{tr}_2\Big(m_{12}(0)\ \overset{2}{S}\Big)$ are defined through the coefficients of R-matrix expansion in the classical limit $R^\hbar_{12}(z) = \hbar^{-1}1_N \otimes 1_N + r_{12}(z) + \hbar\, m_{12}(z) + O(\hbar^2)$ and $r^{(0)}_{12}$ is the coefficient in the expansion $r_{12}(z) = z^{-1}P_{12} + r^{(0)}_{12} + O(z)$, where P_{12} is the permutation operator. Equa-

[1)]e-mail: kantemir.atalikov@yandex.ru; zotov@mi-ras.ru tions (3) are Hamiltonian with the following Hamiltonian function:

$$H^{\rm LL} = \oint dy \Big(cN{\rm tr}\Big(S\,J(S)\Big) - \frac{Nk^2}{2c}\,{\rm tr}\Big(\partial_y S\,\partial_y S\Big) + \\ + kN{\rm tr}\Big(\partial_y S\,E(S)\Big)\Big), \quad S=S(y). \qquad (4)$$

In this paper we use the rational R-matrix calculated in [9]. In the $N=2$ case it reproduces the 11-vertex R-matrix found by I. Cherednik [10]. For $N>2$ all its properties, different possible forms and explicit expressions for the coefficients of expansions near $z=0$ and $\hbar=0$ can be found in [11].

The statement is that by applying the gauge transformation with a certain matrix $G(z)$ we obtain the desired relation (2). Calculations are performed similarly to those in $0+1$ mechanics [12]. As a result we obtain explicit change of variables expressed through elementary symmetric functions σ_k:

$$S_{ij} = \frac{(-1)^{\varrho(j)+1}}{N} \times \\ \times \sum_{m=1}^{N} \frac{(q_m)^{\varrho(i)}(\tilde{p}_m + \frac{k\alpha_{mx}}{\alpha_m}) + \alpha_m^2 \varrho(i)(q_m)^{\varrho(i)-1}}{\prod_{l\neq m}(q_m - q_l)}\,\sigma_{\varrho(j)}(q),$$

$$\tilde{p}_j = p_j - \sum_{l\neq j}^{N} \frac{\alpha_j^2}{q_j - q_l} \qquad (5)$$

(here $\varrho(i) = i-1$ for $i \le N-1$ and $\varrho(i) = N$ for $i=N$) with the properties

$${\rm Spec}(S) = (0,...,0,c), \ \ {\rm rk}(S)=1, \ \ {\rm tr}(S) = c, \ \ S^2 = cS, \qquad (6)$$

where $\alpha_i^2 = kq_{ix} - c$. It can be also verified that the Poisson brackets for $S_{ij}(p,q,c)$ calculated through the canonical brackets for p_i, q_j indeed reproduce the linear Poisson structure, so that (5) is a Poisson map. The Hamiltonian (1) of $1+1$ field Calogero–Moser model coincides with the one (4) for the Landau–Lifshitz equation under the change of variables (5): $H^{\rm LL}[S(p(x),q(x))] = H^{\rm 2dCM}[p(x),q(x)]$.

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