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Higher rank 1 + 1 integrable Landau–Lifshitz field theories from associative Yang–Baxter equation

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We suggest a generalization of the Landau–Lifshitz equation

$$\partial_t S = c_1[S, J(S)] + c_2[S, \partial_x^2 S],$$
$$S = \sum_{k=1}^{3} S_k\sigma_k, \quad J(S) = \sum_{k=1}^{3} S_k J_k \sigma_k, \tag{1}$$

where $\tilde{c}_1$, $\tilde{c}_2$ and J_1, J_2, J_3 are some constants. The periodic boundary conditions $\mathbf{S}(t,x) = \mathbf{S}(t, x+2\pi)$ are assumed. The widely known Sklyanin's result is that this equation is represented in the Zakharov–Shabat (or zero-curvature) form:

$$\partial_t U(z) - \partial_x V(z) + [U(z), V(z)] = 0, \tag{2}$$

$$U(z) = \sum_{k=1}^{3} S_k \sigma_k \varphi_k(z),$$
$$V(z) = \sum_{k=1}^{3} S_k\sigma_k \frac{\varphi_1(z)\varphi_2(z)\varphi_3(z)}{\varphi_k(z)} + \sum_{k=1}^{3} W_k \sigma_k \varphi_k(z), \tag{3}$$

where $\varphi_k(z)$ is a certain set of elliptic functions.

In this paper we propose a construction of $1+1$ integrable Heisenberg–Landau–Lifshitz type equations in the gl_N case. The dynamical variables are matrix elements of $N \times N$ matrix S with the property $S^2 = \mathrm{const} \cdot S$. The Lax pair with spectral parameter is constructed by means of a quantum R-matrix satisfying the associative Yang–Baxter equation. The family of such R-matrices includes the elliptic GL_N Baxter–Belavin R-matrix and its trigonometric and rational degenerations.

Consider expansion of a quantum R-matrix in the classical limit and the expansion of the classical r-matrix near the pole in spectral parameter:

$$R_{12}^{\hbar}(z) = \frac{1}{\hbar} 1_N \otimes 1_N + r_{12}(z) + \hbar\, m_{12}(z) + O(\hbar^2),$$
$$r_{12}(z) = \frac{1}{z} N P_{12} + r_{12}^{(0)} + O(z). \tag{4}$$

Next, define the following linear maps (here $\overset{2}{A} = 1_N \otimes A$ for $A \in \mathrm{Mat}(N, \mathbb{C})$):

$$A \to E(A) = \frac{1}{N} \mathrm{tr}_2 \Big(r_{12}^{(0)} \overset{2}{A} \Big),$$
$$A \to J(A) = \frac{1}{N} \mathrm{tr}_2 \Big(m_{12}(0) \overset{2}{A} \Big). \tag{5}$$

Using R-matrix identities one can show that the Lax pair

$$U(z) = L(S, z) = \frac{1}{N} \mathrm{tr}_2 \Big(r_{12}(z) \overset{2}{S} \Big),$$
$$V(z) = V_1(z) + V_2(z), \tag{6}$$

$$V_1(z) = -c\partial_z L(S, z) + L(SE(S), z) + L(E(S)S, z),$$
$$V_2(z) = -cL(T, z), \tag{7}$$

where $T = -c^{-2}[S, \partial_x S]$, satisfies the Zakharov–Shabat equations identically in spectral parameter and provides the following equations of motion:

$$\partial_t S - \frac{1}{c}[S, \partial_x^2 S] - \partial_x \Big(SE(S) + E(S)S \Big) = \tag{8}$$
$$= 2s_0[S, J(S)] - \frac{1}{c}[S, E([S, \partial_x S])] - \frac{1}{c}[E(S), [S, \partial_x S]],$$

which is the gl_N generalization of the Landau–Lifshitz equation. The derived equation is simplified when the matrix S is of rank 1, i.e. $S = \xi \otimes \psi$ ($S_{ij} = \xi_i \eta_j$). Then the equation takes the form

$$\partial_t S = \frac{1}{c}[S, \partial_x^2 S] + \frac{2c}{N}[S, J(S)] - 2[S, E(\partial_x S)]. \tag{9}$$

The latter one equation can be easily described in the Hamiltonian formalism. Namely, it is the Hamiltonian equation coming from the Poisson brackets

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$\{S_{ij}(x), S_{kl}(y)\} = (S_{kj}(x)\delta_{il} - S_{il}(x)\delta_{kj})\delta(x-y)$ and the Hamiltonian function

$$H = \oint dy\Big(\frac{c}{N}\,\mathrm{tr}\Big(S\,J(S)\Big) - \frac{1}{2c}\,\mathrm{tr}\Big(\partial_y S\,\partial_y S\Big) + \\ + \mathrm{tr}\Big(\partial_y S\,E(S)\Big)\Big), \qquad S = S(y). \tag{10}$$

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